MATHEMATICS IN THE NEW INTERNATIONAL ENCYCLOPEDIA.[1]

By G. A. MILLER, University of Illinois.

The *New International Encyclopedia* has won an eminent position among the American works of reference and is found in a large number of public and private libraries. A second edition has recently appeared in twenty-four large volumes, and the eminence of the men on its editorial staff tends to inspire confidence with respect to accuracy and wise selection of material. The present note does not aim to shake this confidence as regards the useful articles relating to mathematics. On the contrary, it aims to increase the interest in these articles by directing attention to a few desirable corrections and especially to the need of a critical study of some of the statements before accepting them as final authority.

The critical mathematical reader can easily convince himself that certain corrections would greatly improve this standard work. By consulting the article under the word *equation* he will find near the end thereof several surprising statements to the effect that the expressions

$$\sqrt{p^2 - 4q}, \qquad \sqrt{g^2 + 4h^3}$$

are called the discriminants of the two equations

$$x^2 + px + q = 0, \qquad x^3 + 3hx + g = 0$$

respectively. While it is well known that the expression "discriminant of an equation in one unknown" has not always been defined in the same manner by recent noteworthy authors, yet all of these authors seem to agree on the point that such a discriminant is a rational function of the coefficients.[2] The fact that the radical signs noted above are found also in volumes dated 1904 and 1912 respectively of the encyclopedia under consideration makes it more difficult to attribute their appearance in the latest edition to an oversight.

With this clear instance where corrections are needed fresh in mind one may be inclined to approach the rest of this article on the equation in a critical spirit and to observe a considerable number of statements which one would like to change. In fact, the sentence beginning in the eleventh line of this article seems to exhibit too narrow a spirit for a big work. It reads as follows: "The expression $2 + 5 = 7$ expresses an equality, but it is not an equation as the word is technically used in mathematics." If one turns to such a standard work as E. Borel's

[1] This note is a part of a paper read before the Iowa Association of Mathematics Teachers, November 3, 1916.

[2] In the *New Standard Dictionary*, 1915, the following two definitions of the term discriminant are given. "The integral function of the coefficients of an algebraic equation that becomes zero only when the equation has equal roots. The discriminant is equal to the continued product of the squares of all the differences of the roots." Even by means of the quadratic equation $ax^2 + bx + c = 0$ it can easily be verified that these two definitions of the term discriminant are contradictory whenever $a \neq 1$. Two definitions which appear in Webster's *New International Dictionary*, 1916, under the word "discriminant" are contradictory for the same reason.

Die Elemente der Mathematik, translated by P. Stäckel, 1908, one finds on page 7 the following identity

$$145 = 145$$

as an illustrative example of an equation (Gleichung). Many good writers speak of "identical equations" and of "conditional equations," and it would seem that an article in a standard work of reference should recognize this fact.

One does not need to read many lines in the article under consideration before reaching the statement "and in the theory of equations, so called, they [the coefficients] stand for real quantities." It is true that equations with real coefficients usually receive most attention in our elementary works on the theory of equations, but the developments frequently include explicitly the case when the coefficients are complex. In particular, the fundamental theorem of algebra is stated in Dickson's *Elementary Theory of Equations,* 1914, page 47, in the following form: "Every equation with complex coefficients $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0$ has a complex (real or imaginary) root." Hence it would appear that the article in question could be improved by cancelling the statement quoted near the beginning of the present paragraph.

Without trying to direct attention to all the statements in the article under consideration which the critical reader might be inclined to modify, we shall refer to the following, appearing near the end of the first column: "In case there is not a sufficient number of relations given to enable the roots of an equation to be determined, exactly or approximately, the equation is said to be *indeterminate; e. g.*, in the equation $x + 2y = 10$ any of the following pairs of values satisfies the equation: (0, 5), (1, 4.5), (2, 4), (3, 3.5), $\cdots$, (10, 0), (11, − 0.5), $\cdots$." Since such number pairs are called roots of the indeterminate equation in question[1] it is somewhat difficult to see why an indeterminate equation should be characterized as one in which not a sufficient number of relations is given to enable the roots to be determined.

These quotations from a single article may suffice to direct attention to the fact that the reader of the mathematical articles in the *New International Encyclopedia* cannot afford to accept as final all the statements contained therein. It should, however, not be inferred that this particular article, which is also marred by a number of typographical errors, is a fair sample of the mathematical articles in the work under consideration. In view of the importance of the subject treated in this article it seems desirable to endeavor to aid the young reader by directing attention to its shortcomings, especially since such a reader is usually inclined to exercise too little caution in accepting results found in what are commonly regarded as standard works of reference.

In the interesting article on *geometry,* contained in volume 9 of the encyclopedia under consideration, there appears on page 610, column 2, the following perplexing sentence: "Riemann and Helmholtz formulated assumptions for a geometry in space of n-ply manifoldness and with constant curvature and observed that on the sphere, whose curvature is constant and positive, the sum of the

[1] Cf. *Encyclopédie des Sciences Mathématiques*, tome I, Vol. 2, p. 55.

angles of a triangle is less than a straight angle, this characterizing the space of the geometry of Bolyai and Lobachevsky." One might at first be inclined to regard the word "less" in this quotation as due to an oversight and to replace it by the word "greater." If this were done there would evidently be trouble with the rest of the sentence since in the Lobachevsky geometry the sum of the angles of a triangle is actually less than two right angles.

That the article on *determinants,* contained in volume 6 of the work under review, is apt to give an incorrect impression of the fundamental difference between the concepts implied by the terms matrix and determinant, as used by the most careful modern writers, results directly from the following sentences which appear near the beginning of this article: "The first or square form of notation is called the array notation. If there are more columns than rows, the form is called a *matrix.*" Probably the critical reader will also be surprised when he meets the following sentence near the close of the article in question: "The theory as a whole has been most systematically treated by F. Brioschi (1824–97), well known as the editor of the *Annali di Matematica,* whose masterly treatise on determinants is a standard (French and German translations, 1856)." F. Brioschi's little treatise was the first book on determinants if we except the monograph by W. Spottiswoode which appeared a little earlier, and it seems very strange that the theory as a whole should be said to have been most systematically treated in the former of these two works when one bears in mind the more modern and more comprehensive treatises by E. Pascal, G. Kowalewski, and others.

The critical mathematical considerations which have been suggested thus far do not relate to what is commonly understood by mathematical history. In view of the emphasis on historical questions in many of the articles in the encyclopedia under review it may be of interest to suggest also a few considerations relating to modifications along this line. To begin with an unusually strong case we may refer to the fact that in the article on *complex number* it is stated that "the first appearance of the imaginary is found in the *Stereometria* of Heron of Alexandria (third century B. C.)." If we turn to the name "Hero or Heron of Alexandria" in the same encyclopedia we meet the following sentence: "The most recent investigation by Schmidt leads to the conclusion that he may have lived in the first century A. D., but other writers, who, it must be said, have not considered the question so fully, have usually placed him in the first or even in the second century B. C." These two statements clearly imply different dates to be associated with the work of an important Greek mathematician. It may be noted in this connection that the same too early date (third century B. C.) for Heron of Alexandria is also found in the *New Standard Dictionary* (1915) and in the *Century Dictionary* (1914).

Another historical statement which seems to need modification appears in the article on *algebra,* in volume 1, page 402, of our encyclopedia, and is as follows: "It was only after the opening of the nineteenth century that Abel, by the use of the theory of groups discovered by Galois, gave the first satisfactory

proof of the fact, anticipated by Gauss and announced by Ruffini, that it is impossible to express the solution of a general equation as algebraic functions of the coefficients when the degree exceeds the fourth." As the first publication by Galois on the theory of groups appeared after the death of Abel it is difficult to see how his proof could have been based upon the discoveries of Galois. At any rate the sentence as quoted above would naturally lead the reader to think that Abel based his work upon work done earlier by Galois and hence it is unsatisfactory in its present form.

We shall refer here to only one more sentence which seems to convey an incorrect impression in regard to a historical fact of considerable mathematical interest. This sentence appears in the article on the Italian mathematician *Cardan,* volume 4, page 536, and is as follows: "The publication of the *Ars Magna* stimulated mathematical research and hastened the general solution of the biquadratic equation, of which Cardan himself had solved special cases." If we read this statement in the light of the fact that Ferrari's general solution of the biquadratic equation actually appeared in the *Ars Magna* it may possibly have some meaning, but it is evident that the beginner would be apt to draw entirely incorrect conclusions therefrom.

The few modifications which have been suggested could scarcely be supposed to be of general interest to mathematics teachers if they did not relate to an excellent work of reference which is very extensively used by college and university students. It is evidently highly desirable that such works be as clear and accurate as possible. We can scarcely expect that the publishers will make special efforts to attain these ends unless the public actually demands them. Hence publicity given to shortcomings, especially where such publicity tends to the discovery of many other important improvements, seems desirable. Such publicity may also tend to inspire caution in the use of even the most reliable works of reference, a caution which needs to be cultivated on the part of most young mathematicians.

NEW RULES OF QUADRATURE.

By P. J. DANIELL, Rice Institute.

The rules given in this paper are developed from Euler's summation formula.[1] This formula has been used in the past chiefly as a means of converting a series into an integral. Nevertheless it has several advantages as a source from which to obtain rules of quadrature. Three such rules are stated here, and the author believes that the second and third are new, while even the first has not received the attention which it deserves.

Rule 1.

$$\int_a^b y\,dx = h[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n] + \frac{h^2}{12}[y_0' - y_n'] + R_1.$$

[1] Bromwich, *Theory of Infinite Series,* Chap. X, p. 238.